Topology and its Applications 156 (2009) 1620–1629

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# Compact and precompact sets in asymmetric locally convex spaces

S. Cobzaş [1]

*Babeş-Bolyai University, Faculty of Mathematics and Computer Science, 400 084 Cluj-Napoca, Romania*

ARTICLE INFO

*Article history:*
Received 20 May 2008
Received in revised form 9 January 2009
Accepted 9 January 2009

*MSC:*
primary 46A03
secondary 46A50, 46B07, 54E15, 54E25, 54E99

*Keywords:*
Quasi-metric spaces
Quasi-uniform spaces
Spaces with asymmetric norm
Asymmetric locally convex spaces
Compactness
Precompactness
Total boundedness

ABSTRACT

The aim of the present paper is to study precompactness and compactness within the framework of asymmetric locally convex spaces, defined and studied by the author in [S. Cobzaş, Asymmetric locally convex spaces, Int. J. Math. Math. Sci. 2005 (16) (2005) 2585–2608]. The obtained results extend some results on compactness in asymmetric normed spaces proved by [L.M. García-Raffi, Compactness and finite dimension in asymmetric normed linear spaces, Topology Appl. 153 (2005) 844–853], and [C. Alegre, I. Ferrando, L.M. García-Raffi, E.A. Sánchez-Pérez, Compactness in asymmetric normed spaces, Topology Appl. 155 (6) (2008) 527–539].



## 1. Introduction

A quasi-uniformity on a set $X$ is a filter $\mathcal{U}$ in $X \times X$ such that:

(QU1) $\Delta(X) \subset U$, $\forall U \in \mathcal{U}$;
(QU2) $\forall U \in \mathcal{U}$, $\exists V \in \mathcal{U}$, such that $V \circ V \subset U$,

where $\Delta(X) = \{(x,x)\colon\ x \in X\}$ denotes the diagonal of $X$ and, for $M, N \subset X \times X$,

$$M \circ N = \big\{(x,z) \in X \times X\colon\ \exists y \in X,\ (x,y) \in M \text{ and } (y,z) \in N\big\}.$$

If the filter $\mathcal{U}$ satisfies also the condition

(U3) $\forall U,\ U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$,

where

$$U^{-1} = \big\{(y,x) \in X \times X\colon\ (x,y) \in U\big\},$$

*E-mail address:* scobzas@math.ubbcluj.ro.
[1] Supported by Grant CEEX-06-11-96.

0166-8641/$ – see front matter 
doi:10.1016/j.topol.2009.01.004

then $\mathcal{U}$ is called a *uniformity* on $X$. The sets in $\mathcal{U}$ are called *entourages*. A *base* for the uniformity $\mathcal{U}$ is a subset $\mathcal{B}$ of $\mathcal{U}$ such that for every $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \subset U$. A *subbase* for $\mathcal{U}$ is a subset $\mathcal{D}$ of $\mathcal{U}$ such that every $U \in \mathcal{U}$ contains a finite intersection of sets in $\mathcal{D}$.

For $U \in \mathcal{U}$, $x \in X$ and $Z \subset X$ put

$$U(x) = \{y \in X:\ (x, y) \in U\} \quad \text{and} \quad U[Z] = \bigcup \{U(z):\ z \in Z\}.$$

A quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on $X$ for which the family of sets

$$\{U(x):\ U \in \mathcal{U}\}$$

is a base of neighborhoods of the point $x \in X$. The lack of the symmetry, i.e., the omission of the axiom (U3), makes the theory of quasi-uniform spaces to differ drastically from that of uniform spaces, mainly in what concerns completeness, compactness and total boundedness, see [7,8,17,23,24,30,34]. A short survey about these questions is given in [4]. An account of the theory of quasi-metric and quasi-uniform spaces up to 1982 is given in the book by Fletcher and Lindgren [9]. The survey papers by Künzi [18–22] are good guides for subsequent developments. Another book on quasi-uniform spaces is [26].

An asymmetric norm on a real vector space $X$ is a functional $p : X \to [0, \infty)$ satisfying the conditions:

(AN1) $p(x) = p(-x) = 0 \Rightarrow x = 0$;
(AN2) $p(\alpha x) = \alpha p(x)$;
(AN3) $p(x + y) \leqslant p(x) + p(y)$,

for all $x, y \in X$ and $\alpha \geqslant 0$. If $p$ satisfies only the conditions (AN2) and (AN3), then it is called an *asymmetric seminorm*.

The conjugate of $p$ is the seminorm $\bar{p} : X \to [0, \infty)$ defined by $\bar{p}(x) = p(-x)$, $x \in X$. The functional $\tilde{p}(x) = \max\{p(x), \bar{p}(x)\}$, $x \in X$, is a (symmetric) seminorm on $X$. If $p$ is an asymmetric norm, then $\tilde{p}$ is a norm. The following inequalities hold for every $x, y \in X$,

$$\left|p(x) - p(y)\right| \leqslant \tilde{p}(x - y) \quad \text{and} \quad \left|\bar{p}(x) - \bar{p}(y)\right| \leqslant \tilde{p}(x - y).$$

A quasi-metric on a set $X$ is a mapping $\rho : X \times X \to [0, \infty)$ satisfying the conditions:

(QM1) $\rho(x, y) = \rho(y, x) = 0 \Leftrightarrow x = y$;
(QM2) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$,

for all $x, y, z \in X$. If the mapping $\rho$ satisfies only the conditions $\rho(x, x) = 0$, $x \in X$, and (QM2), then it is called a *quasi-semimetric*.

If $p$ is an asymmetric norm (seminorm) on a vector space $X$, then $\rho(x, y) = p(y - x)$, $x, y \in X$, is a quasi-metric (respectively a quasi-semimetric) on $X$.

A closed, respectively open, ball in a quasi-semimetric space is defined by

$$B_\rho(x, r) = \{y \in X:\ \rho(x, y) \leqslant r\}, \qquad B'_\rho(x, r) = \{y \in X:\ \rho(x, y) < r\},$$

for $x \in X$ and $r > 0$. In the case of an asymmetric seminorm $p$, one denotes by $B_p(x, r)$, $B'_p(x, r)$ the corresponding balls and by $B_p = B_p(0, 1)$, $B'_p = B'_p(0, 1)$, the unit balls. In this case the following equalities hold

$$B_p(x, r) = x + rB_p \quad \text{and} \quad B'_p(x, r) = x + rB'_p.$$

The family of sets $B'_\rho(x, r)$, $r > 0$, is a base of neighborhoods of the point $x \in X$ for the topology $\tau_\rho$ on $X$ generated by the quasi-metric $\rho$. The family $B_\rho(x, r)$, $r > 0$, of closed balls is also a neighborhood base at $x$ for $\tau_\rho$.

The theory of asymmetric normed spaces has been developed in a series of papers [2,4,5,11–13,27,28,31,32], following ideas from the theory of (symmetric) normed spaces and emphasizing similarities as well as differences between the symmetric and the asymmetric case. The developed results found some nontrivial applications to complexity analysis, see, e.g., [14,29,33].

Let $X$ be a real vector space and $P$ a family of asymmetric seminorms on $X$. Without restricting the generality we can suppose that the family $P$ is directed, that is for every $p_1, p_2 \in P$ there is $p \in P$ such that $p_i \leqslant p$, $i = 1, 2$, the order being the pointwise order. One defines a topology $\tau(P)$ on $X$ by

$$V \in \mathcal{V}(x) \quad \Longrightarrow \quad \exists p \in P,\ \exists r > 0, \qquad B'_p(x, r) \subset V,$$

where $\mathcal{V}(x)$ denotes the family of neighborhoods of the point $x \in X$. Asking that $B_p(x, r) \subset V$ for some $p \in P$ and $r > 0$ one obtains the same topology.

It follows that $\tau(P)$ is a translation invariant topology on $X$, that is

$$V \in \mathcal{V}(x) \quad \Longrightarrow \quad \exists U \in \mathcal{V}(0), \qquad V = x + U.$$

This implies that the addition $+ : X \times X \to X$ is continuous, but the multiplication by scalars $\cdot : \mathbb{R} \times X \to X$ need not be continuous.

A directed family $P$ of asymmetric seminorms on a vector space $X$ generates a quasi-uniformity $\mathcal{U}_P$ having as a base of entourages of the diagonal the sets

$$U_{p,\epsilon} = \big\{(x, y) \in X \times X\colon\ p(y - x) < \epsilon\big\}, \quad p \in P,\ \epsilon > 0.$$

The family

$$\bar{U}_{p,\epsilon} = \big\{(x, y) \in X \times X\colon\ p(y - x) \leqslant \epsilon\big\}, \quad p \in P,\ \epsilon > 0,$$

generates the same quasi-uniformity.

Since $U_{p,\epsilon}(x) = B'_p(x, \epsilon)$ and $\bar{U}_{p,\epsilon}(x) = B_p(x, \epsilon)$, it follows that the quasi-uniformity $\mathcal{U}_P$ generates the topology $\tau(P)$ on $X$.

In [3] asymmetric locally convex spaces were defined and some of their basic properties were proved: continuity of linear mapping in terms of a semi-Lipschitz condition, linear functionals and weak topologies, the dual space, separation of convex sets, extreme points and the Krein–Milman theorem.

For a family $P$ of asymmetric seminorms on a vector space $X$ put

$$\bar{P} = \{\bar{p}\colon\ p \in P\} \quad \text{and} \quad \tilde{P} = \{\tilde{p}\colon\ p \in P\}. \tag{1.1}$$

Along this paper, $P$ will always stand for a directed family of asymmetric seminorms on a vector space.

The relations between the topologies generated by $P$, $\bar{P}$ and $\tilde{P}$ are explained in the following proposition.

**Proposition 1.1.** *Let $P$ be a directed family of asymmetric seminorms on a real vector space $X$. Then*

(1) *For every $p \in P$, $x \in X$ and $\epsilon > 0$,*

$$B'_{\tilde{p}}(x, \epsilon) = B'_p(x, \epsilon) \cap B'_{\bar{p}}(x, \epsilon) \quad \textit{and} \quad B_{\tilde{p}}(x, \epsilon) = B_p(x, \epsilon) \cap B_{\bar{p}}(x, \epsilon).$$

(2) *Any $\tau(P)$-open set is $\tau(\tilde{P})$-open and any $\tau(\bar{P})$-open set is $\tau(\tilde{P})$-open, that is $\tau(P) \subset \tau(\tilde{P})$ and $\tau(\bar{P}) \subset \tau(\tilde{P})$. The same inclusions hold for the corresponding closed sets.*
(3) *Any $\tau(P)$-continuous (or $\tau(\bar{P})$-continuous) mapping $f$ from $X$ to a topological space $T$ is $\tau(\tilde{P})$-continuous.*
(4) *A ball $B'_p(x, r)$ is $\tau(P)$-open. A ball $B_p(x, r)$ is $\tau(\bar{P})$-closed and it could be not $\tau(P)$-closed.*

**Proof.** The assertions (1) and (2) are immediate and (3) is a consequence of (2), so we only need to prove (4).

For $y \in B'_p(x, r)$ let $r' := r - p(y - x) > 0$. Since $p(z - y) < r'$ implies $p(z - x) \leqslant p(z - y) + p(y - x) < r' + p(y - x) = r$ it follows $B'_p(y, r') \subset B'_p(x, r)$.

To prove that $B_p(x, r)$ is $\tau(\bar{P})$-closed let $y \in X \setminus B_p(x, r)$. Then $r' := p(y - x) - r > 0$ and $B'_{\bar{p}}(y, r') \subset X \setminus B_p(x, r)$. Indeed, if there would exist an element $z \in B_p(x, r) \cap B'_{\bar{p}}(y, r')$, then one obtains the contradiction

$$p(y - x) \leqslant p(y - z) + p(z - x) = \bar{p}(y - z) + p(z - x) < r' + r = p(y - x).$$

Consequently, $X \setminus B_p(x, r)$ is $\tau(\bar{P})$-open and so $B_p(x, r)$ is $\tau(\bar{P})$-closed. □

**Example 1.2.** In $\mathbb{R}$ with the upper topology $\tau_u$, where $u(x) = \max\{x, 0\}$, $x \in \mathbb{R}$, we have $B_u(0, 1) = (-\infty; 1]$ and $\mathbb{R} \setminus B_u(0, 1) = (1; +\infty)$ is $\tau_{\bar{u}}$-open, but not $\tau_u$-open.

**Remark 1.3.** As a space with two topologies, $\tau(P)$ and $\tau(\bar{P})$, an asymmetric LCS is also a bitopological space in the sense of Kelly [16].

The topology $\tau(P)$ is not Hausdorff in general. For instance, the space $(\mathbb{R}, \tau_u)$ from Example 1.2 is $T_0$ but not $T_1$. Indeed, if $x < y$, then for $\epsilon := (y - x)/2$, $y \notin (-\infty, x + \epsilon)$, but every neighborhood of $y$ contains $x$.

We have:

**Proposition 1.4.** *Let $P$ be a directed family of asymmetric seminorms on a real vector space $X$.*

(1) *The topology $\tau(P)$ is $T_0$ if and only if for every $x \in X$, $p(x) = p(-x) = 0$ for all $p \in P$ implies $x = 0$.*
(2) *The topology $\tau(P)$ is $T_1$ if and only if for every $x \neq 0$ there exists $p \in P$ such that $p(x) > 0$.*

**Proof.** (1) Suppose that $\tau(P)$ is $T_0$ and let $x \neq 0$ be an element of $X$. Then there exist $p \in P$ and $r > 0$ such that $x \notin B'_p(0,r)$, implying $p(x) \geqslant r > 0$, or there exist $q \in P$ and $s > 0$ such that $0 \notin B'_q(x,s)$, implying $q(-x) = q(0-x) \geqslant s > 0$.

Conversely, suppose that the condition on $P$ holds and let $x, y \in X$, $x \neq y$. By hypothesis, there exists $p \in P$ such that $p(x-y) > 0$ or $p(y-x) > 0$. Then $x \notin B'_p(y,r)$ in the first case, with $r := p(x-y)$, while $y \notin B_p(x,r')$ in the second one, with $r' = p(y-x)$.

(2) Suppose that $\tau(P)$ is $T_1$ and let $x \in X$, $x \neq 0$. By hypothesis, there exist $p \in P$ and $r > 0$ such that $x \notin B'_p(0,r)$, implying $p(x) \geqslant r > 0$.

Conversely, suppose that for every $x \neq 0$ in $X$ there exists $p \in P$ with $p(x) > 0$. If $x, y \in X$, $x \neq y$, then there exists $p, q \in P$ such that $r := p(x-y) > 0$ and $r' := q(y-x) > 0$, implying $x \notin B_p(y,r)$ and $y \notin B_q(x,r')$. □

The characterization of Hausdorff separation is more complicated and was done in [12] in the case of asymmetric normed spaces and extended to asymmetric LCS in [3]. For an asymmetric seminorm $p : X \to \mathbb{R}$ let $p^\diamond : X \to \mathbb{R}$ be defined by

$$p^\diamond(x) = \inf\{p(y) + p(y-x):\ y \in X\}, \quad x \in X.$$

It follows that $p^\diamond$ is a (symmetric) seminorm on $X$ and $p^\diamond \leqslant p$. Moreover, $p^\diamond$ is the greatest seminorm majorized by $p$.

**Proposition 1.5.** *([3,12]) Let $X$ be a real vector space, $P$ a family of asymmetric seminorms on $X$ and $P^\diamond = \{p^\diamond:\ p \in P\}$.*

*The topology $\tau(P)$ is Hausdorff (or $T_2$) if and only if for every $x \in X$, $x \neq 0$, there exists $p \in P$ such that $p^\diamond(x) > 0$.*

*In particular, the topology $\tau_p$ generated by a seminorm $p$ on a vector space $X$ is Hausdorff if and only if $p^\diamond(x) > 0$ for every $x \neq 0$, that is, if and only if $p^\diamond$ is a norm on $X$.*

A topological space $(T, \tau)$ is called *regular* or $T_3$ if it is $T_1$ and for each $t \in T$ and each closed subset $S$ of $T$ not containing $t$ there are disjoint open subsets $U, V$ of $T$ such that $t \in U$ and $S \subset V$. In other words a point and a closed set not containing it can be separated by open sets. The space $T$ is called *completely regular*, or *Tychonoff*, or $T_{3\frac{1}{2}}$, if for every $t \in T$ and every closed subset $S$ of $T$ not containing $t$ there is a continuous function $f : T \to [0;1]$ such that $f(t) = 1$ and $f(s) = 0$ for each $s \in S$. A strong result in functional analysis asserts that a $T_0$ topological vector space is completely regular (see [25, Theorem 2.2.14]). The example of the space $\mathbb{R}$ with the upper topology $\tau_u$, which is $T_0$ but not $T_1$, shows that this result is not longer true in asymmetric normed spaces. Proposition 1.8 below shows that a finite dimensional asymmetric LCS which is $T_1$ is, in fact, a Hausdorff topological vector space, so it is also completely regular. I do not know if a similar result holds in the infinite dimensional case.

It is well known that any finite dimensional Hausdorff topological vector space $X$ is topologically isomorphic with the Euclidean $\mathbb{R}^m$, where $m = \dim X$. García-Raffi [10] proved that the result still holds for finite dimensional $T_1$ asymmetric normed spaces. Following the ideas from [10] we shall extend in this paper this result to asymmetric LCS, a setting which requires nets instead of sequences (see, e.g., [6] or [15]). A net in a topological space $X$ is a mapping $\varphi : I \to X$, where $I$ is a directed set. A net is also denoted by $(x_i:\ i \in I)$, where $x_i = \varphi(i)$. A subset $J$ of $I$ is called *cofinal* in $I$ provided for every $i \in I$ there exists $j \in J$ with $i \leqslant j$. One says that a net $\psi : J \to X$ is a *subnet* of the net $\varphi : I \to X$ if there exists a monotone mapping $\lambda : J \to I$ (i.e., $j_1 \leqslant_J j_2$ implies $\lambda(j_1) \leqslant_I \lambda(j_2)$) such that $\psi = \varphi \circ \lambda$ and $\lambda(J)$ is cofinal in $I$. A subnet of the net $(x_i:\ i \in I)$ can be denoted also by $(x_{\lambda(j)}:\ j \in J)$. If $J$ is a cofinal subset of $I$, then $(x_j:\ j \in J)$ is a subnet of $(x_i:\ i \in I)$. It is clear that, if $J_1$ is a cofinal subset of the directed set $I$ and $J_2$ is a cofinal subset of $J_1$, then $J_2$ is also a cofinal subset of $I$.

The following result is probably well-known, but for the sake of reader convenience we include a proof.

**Lemma 1.6.** *Let $(I, \leqslant)$ be a directed set. If $I = J_1 \cup \cdots \cup J_m$, where $J_k$ are nonempty subsets of $I$, $k = 1, \ldots, m$, then at least one of the sets $J_k$ is cofinal in $I$.*

**Proof.** If $J_1$ is a cofinal subsets of $I$, then we are done. If $J_1$ is not cofinal in $I$, then there exists $i_1 \in I$ such that there is no $j \in J_1$ with $i_1 \leqslant j$. Putting

$$I_1 = \{i \in I:\ i \neq i_1 \text{ and } i_1 \leqslant i\},$$

it follows $I_1 \subset J_2 \cup \cdots \cup J_m$. We distinct two cases.

(I) $I_1 = \emptyset$.

In this case $i_1$ is the greatest element of $I$. Indeed for $i \in I$ there exists $i_2 \in I$ such that $i_1 \leqslant i_2$ and $i \leqslant i_2$. By hypothesis $i_2 = i_1$, so that $i \leqslant i_1$. If $k \in \{1, \ldots, m\}$ is such that $i_1 \in J_k$, then $J_k$ is a cofinal subset of $I$.

(II) $I_1 \neq \emptyset$.

In this case the set $I_1$ is cofinal in $I$. Indeed, since $I_1$ is nonempty there exists an element $j_1 \in I_1$. If $i \in I$ is arbitrary, then there exists $j \in I$ such that $i \leqslant j$ and $j_1 \leqslant j$, implying $j \in I_1$ and $i \leqslant j$. Since $I_1$ is contained in $J_2 \cup \cdots \cup J_m$, it follows that $J_2 \cup \cdots \cup J_m$ is also a cofinal subset of $I$.

Repeating the argument with $J_2 \cup \dots \cup J_m$ instead of $I$ and continuing in this manner, we get that some $J_k$ is a cofinal subset of $I$. $\square$

An immediate consequence of this lemma is the following proposition.

**Proposition 1.7.** *If $Y$ is a $\tau(P)$-closed subset of an asymmetric LCS $(X, P)$, then $Z + Y$ is $\tau(P)$-closed for every finite subset $Z$ of $X$.*

**Proof.** Let $Z = \{z_1, \dots, z_m\}$ and let $(x_i\colon\ i \in I)$ be a net in $Z + Y$ which is $\tau(P)$-convergent to some $x \in X$. Then for every $i \in I$ there exists $k_i \in \{1, \dots, m\}$ and $y_i \in Y$ such that $x_i = z_{k_i} + y_i$. If for every $k \in \{1, \dots, m\}$ the set $J_k$ is defined by $J_k = \{i \in I\colon\ x_i = z_k + y_i\}$, then $I = J_1 \cup \dots \cup J_m$, so that, by Lemma 1.6, there exists $k \in \{1, \dots, m\}$ such that the set $J_k$ is cofinal in $I$. But then $x_j = z_k + y_j$, $j \in J_k$, is a subnet of the net $(x_i\colon\ i \in I)$, so it is $\tau(P)$-convergent to $x$. It follows that for every $p \in P$,

$$p\big(y_j - (x - z_k)\big) = p(z + y_j - x) \to 0, \quad j \in J_k,$$

which shows that $(y_j\colon\ j \in J_k)$ is $\tau(P)$-convergent to $x - z_k$. Since $Y$ is $\tau(P)$-closed, $x - z_k$ belongs to $Y$, implying $x \in z_k + Y \subset Z + Y$. $\square$

The isomorphism result is the following.

**Proposition 1.8.** *Let $(X, P)$ be an asymmetric LCS whose topology $\tau(P)$ is $T_1$. If $X$ is finite dimensional with* $\dim X = m$*, then it is topologically isomorphic with the Euclidean space $\mathbb{R}^m$.*

The proposition will be an immediate consequence of the following lemma, proved in the case of an asymmetric normed space in [10].

**Lemma 1.9.** *Let $(X, P)$ be an asymmetric LCS of finite dimension $m \geqslant 1$ with basis $e_1, \dots, e_m$ and let $x_i = \alpha_{1,i}e_1 + \dots + \alpha_{m,i}e_m$, $i \in I$, be a net in $X$.*

(1) *If for every $k \in \{1, \dots, m\}$ the net $(\alpha_{k,i})$ converges in $\mathbb{R}$ to some $\alpha_k \in \mathbb{R}$, then the net $(x_i)$ converges to $x = \alpha_1 e_1 + \dots + \alpha_m e_m$ with respect to the topology $\tau(P)$.*
(2) *If the topology $\tau(P)$ is $T_1$ and the net $(x_i)$ converges with respect to $\tau(P)$ to $x = \alpha_1 e_1 + \dots + \alpha_m e_m$, then the net $(\alpha_{k,i})$ converges in $\mathbb{R}$ to $\alpha_k$ for every $k \in \{1, \dots, m\}$.*

**Proof.** (1) For any $p \in P$,

$$p(x_i - x) = p\left(\sum_{k=1}^{m}(\alpha_{k,i} - \alpha_k)e_k\right) \leqslant \tilde{p}\left(\sum_{k=1}^{m}(\alpha_{k,i} - \alpha_k)e_k\right) \leqslant \sum_{k=1}^{m}|\alpha_{k,i} - \alpha_k|\tilde{p}(e_k) \to 0, \quad \text{for } i \in I.$$

Here $\tilde{p}(x) = \max\{p(x), p(-x)\}$ denotes the symmetric norm associated to $p$.

(2) Suppose, by contradiction, that $p(x_i) \to 0$ for every $p \in P$, but at least one of the nets $(\alpha_{k,i})$, say $(\alpha_{1,i})$, does not converge to 0 in $\mathbb{R}$. Then there exists $\epsilon > 0$ such that for every $i \in I$ there exists $j \in I$, $j \geqslant i$, such that $|\alpha_{1,j}| \geqslant \epsilon$. This implies that the set $J = \{j \in I\colon\ |\alpha_{1,j}| \geqslant \epsilon\}$ is cofinal in $I$, and, consequently, $(x_j\colon\ j \in J)$ is a subnet of $(x_i\colon\ i \in I)$, so it also converges to 0 with respect to $\tau(P)$. It follows also that $M_j := \max\{|\alpha_{1,j}|\colon\ 1 \leqslant k \leqslant m\} \geqslant \epsilon$ for all $j \in J$.

If $y_j := M_j^{-1}x_j$, $j \in J$, then

$$p(y_j) = \frac{1}{M_j}p(x_j) \leqslant \frac{1}{\epsilon}p(x_j) \to 0, \quad j \in J.$$

Writing $y_j = \beta_{1,j}e_1 + \dots + \beta_{m,j}e_m$, $j \in J$, it follows that for every $j \in J$, $|\beta_{k,j}| \leqslant 1$, $k = 1, \dots, m$, and at least one of the numbers $\beta_{k,j}$ has modulus one.

If $J_k := \{j \in J\colon\ |\beta_{k,j}| = 1\}$, $k = 1, \dots, m$, then, by Lemma 1.6, at least one of the sets $J_k$, say $J_1$, is cofinal in $J$. By the same lemma, one of the sets $J_1^s = \{j \in J_1\colon\ \beta_{1,j} = (-1)^s\}$, $s = 1, 2$, is cofinal in $J_1$. Denote it by $A_1$. Since the net $(\beta_{2,j}\colon\ j \in A_1)$ is bounded, there exists a subnet $(\beta_{2,\alpha}\colon\ \alpha \in A_2)$ of it converging to some $\beta_2 \in \mathbb{R}$. Similarly, the bounded net $(\beta_{3,\alpha}\colon\ \alpha \in A_2)$ contains a subnet $(\beta_{3,\alpha}\colon\ \alpha \in A_3)$ converging to some $\beta_3 \in \mathbb{R}$. Continuing in this way we obtain the subnets $(\beta_{k,\alpha}\colon\ \alpha \in A_m)$ converging to $\beta_k \in \mathbb{R}$ for every $k = 1, \dots, m$, with $|\beta_1| = 1$. Let $z_\alpha = \beta_{1,\alpha}e_1 + \dots + \beta_{m,\alpha}e_m$, $\alpha \in A_m$, and $z := \beta_1 e_1 + \dots + \beta_m e_m \neq 0$. By the first part of the lemma, the net $(-z_\alpha)$ is $\tau(P)$-convergent to $-z$, which is equivalent to $p(-z_\alpha + z) \to 0$ for every $p \in P$. Since $\tau(P)$ is $T_1$, there exists $p_0 \in P$ such that $p_0(z) > 0$. It follows

$$0 < p_0(z) \leqslant p_0(-z_\alpha + z) + p_0(z_\alpha),$$

in contradiction to the fact that $p_0(z_\alpha) \to 0$. $\square$

## 2. Precompact sets

A subset $Y$ of a quasi-uniform space $(X,\mathcal{U})$ is called *precompact* if for every $U \in \mathcal{U}$ there exists a finite subset $Z$ of $Y$ such that $Y \subset U[Z]$. The set $Y$ is called *totally bounded* if for every $U$ there exists a finite family $A_1,\dots,A_n$ of subsets of $Y$ such that $A_i \times A_i \subset U$, $i=1,\dots,n$, and $Y \subset \bigcup_{i=1}^n A_i$. Note that the total boundedness with respect to $\mathcal{U}$ is equivalent to the total boundedness with respect to the associated uniformity $\mathcal{U}_s$.

If in the above definition of precompactness one asks that the finite set $Z$ be contained in $X$, then one obtains the notions of *outside precompactness* considered in [1]. Obviously, the precompactness implies the outside precompactness, but the reverse implication is not true, even in asymmetric normed spaces, [1]. In uniform spaces the total boundedness, the precompactness and the outside precompactness agree, and a set is compact if and only if it is totally bounded and complete.

If $p$ is an asymmetric seminorm on a vector space $X$, we say that a subset $Y$ of $X$ is *p-precompact* if for every $\epsilon > 0$ there exists a finite subset $Z$ of $Y$ such that

$$Y \subset \bigcup_{z\in Z} B'_p(z,\epsilon). \tag{2.1}$$

If for every $\epsilon > 0$ there exists a finite subset $Z$ of $X$ such that (2.1) holds, then the set $Y$ is called *outside p-precompact*. One obtains an equivalent notion if one asks that $Y$ is covered by the family $B_p(z,\epsilon)$, $z\in Z$, of closed balls. The set $Z$ is called also a $(p,\epsilon)$-net for $Y$ (in both cases).

A subset of an asymmetric LCS $(X,P)$ is called *precompact* if it is precompact with respect to the quasi-uniformity $\mathcal{U}_P$. The following proposition contains a useful characterization of precompactness in asymmetric LCS in terms of seminorms. The proof follows immediately from the definition of the quasi-uniformity $\mathcal{U}_P$ (the fact that $U_{p,\epsilon}(x) = B'_p(x,\epsilon)$).

**Proposition 2.1.** *A subset $Y$ of an asymmetric LCS $(X,P)$ is (outside) precompact if and only if it is (outside) $p$-precompact for every $p\in P$.*

Based on this proposition, the method from [1, Proposition 4] can be adapted to obtain the following relation between precompactness and outside precompactness.

**Proposition 2.2.** *Let $(X,P)$ be an asymmetric LCS. A subset $Y$ of $X$ is $P$-precompact if and only if for every $p\in P$ and every $\epsilon > 0$ and there exists a finite subset $\{x_1,\dots,x_n\}$ of $X$ such that $Y \subset \bigcup_{i=1}^n B'_p(x_i,\epsilon)$ and $Y \cap B'_{\bar p}(x_i,\epsilon) \neq \emptyset$ for every $i \in \{1,\dots,n\}$.*

As a consequence of Proposition 1.1, one obtains the following relations between various notions of compactness and precompactness. A subset $Y$ of an asymmetric LCS $(X,P)$ is called *$P$-bounded* provided $\sup\{p(y):\ y\in Y\} < \infty$ for every $p\in P$. This is equivalent to the fact that it is absorbed by every $\tau(P)$-neighborhood of 0, that is for every $\tau(P)$-neighborhood $V$ of 0 there exists $\lambda > 0$ such that $\lambda Y \subset V$, or, in other words, $Y$ is topologically bounded.

**Proposition 2.3.** *Let $(X,P)$ be an asymmetric LCS and $Y$ a subset of $X$.*

(1) *If the set $Y$ is $\tilde P$-precompact, then it is $P$-precompact and $\bar P$-precompact. The same is true for the outside precompactness.*
(2) *If the set $Y$ is $\tau(\tilde P)$-compact, then it is $\tau(P)$-compact and $\tau(\bar P)$-compact.*
(3) *The outside $P$-precompact subsets of $X$ are $P$-bounded. In particular, the $P$-precompact subsets of $X$ are $P$-bounded as well.*
(4) *A subset of $X$ is $P$-precompact if and only if its $\tau(\bar P)$-closure is $P$-precompact. The same is true for outside $P$-precompactness.*

**Proof.** (1) For $\epsilon > 0$ and $p\in P$ there exists a finite subset $\{y_1,\dots,y_n\}$ of $Y$ such that $Y \subset \bigcup_{i=1}^n B'_{\tilde p}(y_i,\epsilon)$, and so $Y$ is $P$-precompact. Since $B'_{\tilde p}(y_i,\epsilon) \subset B'_p(y_i,\epsilon)$, $i=1,\dots,n$, it follows $Y \subset \bigcup_{i=1}^n B'_p(y_i,\epsilon)$. Similarly, $B'_{\tilde p}(y_i,\epsilon) \subset B'_{\bar p}(y_i,\epsilon)$, $i = 1,\dots,n$, implies $Y \subset \bigcup_{i=1}^n B'_{\bar p}(y_i,\epsilon)$, showing that $Y$ is $\bar P$-precompact. The case of outside precompactness can be treated exactly in the same way.

(2) Let $\{G_i:\ i\in I\}$ be a covering of $Y$ with $\tau(P)$-open sets. Since $\tau(P) \subset \tau(\tilde P)$ and $Y$ is $\tau(\tilde P)$-compact, there exists a finite subset $J$ of $I$ such that $Y \subset \bigcup_{j\in J} G_j$, proving the $\tau(P)$-compactness of $Y$. Similarly, the inclusion $\tau(\bar P) \subset \tau(\tilde P)$ implies the $\tau(\bar P)$-compactness of the $\tau(\tilde P)$-compact set $Y$.

(3) For $p\in P$ there exists a finite subset $\{x_1,\dots,x_n\}$ of $X$ such that $Y \subset \{x_1,\dots,x_n\} + B_p(0,1)$, implying $p(y) \leqslant \max\{p(x_i):\ 1\leqslant i\leqslant n\} + 1$ for every $y\in Y$.

(4) We give a proof different from that in [1]. Suppose first that $Y$ is $P$-precompact and show that $Z = \tau(\bar P)\text{-cl}\, Y$ is also $P$-precompact, which is equivalent to the fact that $Z$ is $p$-precompact for every $p\in P$.

Let $p\in P$ and $\epsilon > 0$. Since $Y$ is $p$-precompact there exists $y_1,\dots,y_n \in Y$ such that

$$Y \subset \bigcup_{i=1}^n B_p(y_i,\epsilon). \tag{2.2}$$

By Proposition 1.1(4) the ball $B_p(0,\epsilon)$ is $\tau(\bar{P})$-closed, so that, by Proposition 1.7, the set

$$\bigcup_{i=1}^{n} B_p(y_i,\epsilon) = \{y_1,\dots,y_n\} + B_p(0,\epsilon)$$

is also $\tau(\bar{P})$-closed. But then, the inclusion (2.2) implies

$$\tau(\bar{P})\text{-}\operatorname{cl} Y \subset \bigcup_{i=1}^{n} B_p(y_i,\epsilon).$$

Conversely, suppose that $Z = \tau(\bar{P})$-cl $Y$ is $P$-precompact and prove that $Y$ is $P$-precompact.

For $p \in P$ and $\epsilon > 0$ there exist $z_1,\dots,z_n \in Z$ such that

$$Z \subset \bigcup_{i=1}^{n} B'_p(z_i,\epsilon). \tag{2.3}$$

For every $i \in \{1,\dots,n\}$ there exists $y_i \in Y \cap B'_{\bar{p}}(z_i,\epsilon)$, that is an $y_i \in Y$ such that $\bar{p}(y_i - z_i) < \epsilon$, or, equivalently, $p(z_i - y_i) < \epsilon$.

Let $y \in Y \subset Z$. By (2.3) there exists $j \in \{1,\dots,n\}$ such that $y \in B'_p(z_j,\epsilon)$. But then

$$p(y - y_j) \leqslant p(y - z_j) + p(z_j - y_j) < 2\epsilon,$$

showing that $Y \subset \bigcup_{i=1}^{n} B'_p(y_i, 2\epsilon)$.

In the case of outside precompactness, a subset of an outside precompact set is also outside precompact, so the outside precompactness of the $\tau(\bar{P})$-closure of $Y$ implies the outside precompactness of $Y$. The reverse implication can be proved exactly as in the case of the precompactness. □

**Remark 2.4.** In the case of asymmetric normed spaces, the result from the assertion 4 of the above proposition was proved by García-Raffi [10, Proposition 9].

The following proposition, a consequence of Proposition 1.8, extends to the asymmetric case a well-known characterization of finite dimensional topological vector spaces. For the proof we need a lemma.

**Lemma 2.5.** *Let $(X,P)$ be an asymmetric LCS, $Q \subset P$ and $D \subset \mathbb{R}$ such that the family $\{B_q(0,r)\colon q \in Q,\ r \in D\}$ is a basis of $\tau(P)$-neighborhoods of* 0. *Then*

$$\tau(P)\text{-}\operatorname{cl} Y = \bigcap \big\{ Y + B_{\bar{q}}(0,r)\colon q \in Q,\ r \in D \big\}, \tag{2.4}$$

*for every subset $Y$ of $X$.*

**Proof.** Let $x \in \tau(P)$-cl $Y$, $q \in Q$ and $r \in D$. Then $x + B_q(0,r)$ is a $\tau(P)$-neighborhood of $x$, so that $Y \cap (x + B_q(0,r)) \neq \emptyset$, implying $x + u = y$, for some $u \in B_q(0,r)$ and $y \in Y$. But

$$u \in B_q(0,r) \iff q(u) \leqslant r \iff \bar{q}(-u) \leqslant r \iff -u \in B_{\bar{q}}(0,r),$$

so that $x = y - u \in Y + B_{\bar{q}}(0,r)$.

Conversely, suppose that $x$ belongs to the intersection from the right-hand side of the equality (2.4). For a $\tau(P)$-neighborhood $V_0$ of 0, let $q \in Q$ and $r \in D$ be such that $B_q(0,r) \subset V_0$. By hypothesis, $x = y + v$ for some $y \in Y$ and $v \in B_{\bar{q}}(0,r)$, which, as above, implies that

$$y = x - v \in x + B_q(0,r) \subset x + V_0.$$

Consequently, $(x + V_0) \cap Y \neq \emptyset$, showing that $x \in \tau(P)$-cl $Y$. □

**Proposition 2.6.** *Let $(X,P)$ be an asymmetric LCS whose topology $\tau(P)$ is $T_1$. Then $X$ is finite dimensional if and only if there exists an outside $P$-precompact $\tau(P)$-neighborhood of* 0.

**Proof.** *Necessity*. If $\dim X = m$, then, by Proposition 1.8, it is isomorphic, algebraically and topologically, with the Euclidean space $\mathbb{R}^m$. Let $\Phi : \mathbb{R}^m \to X$ be an isomorphism. Since the closed unit ball $B_{\mathbb{R}^m}$ is a compact neighborhood of $0 \in \mathbb{R}^m$, its image by $\Phi$ will be a $\tau(P)$-compact neighborhood of $0 \in X$ which will be outside $P$-precompact.

*Sufficiency.* Let $U = B_{p_0}(0,r_0)$ be an outside $P$-precompact $\tau(P)$-neighborhood of 0. Then there exists a finite subset $\{x_1,\dots,x_n\}$ of $X$ such that

$$U \subset \{x_1, \dots, x_n\} + \frac{1}{2}U$$

implying

$$U \subset Z + \frac{1}{2}U, \tag{2.5}$$

where $Z = \mathrm{lin}\{x_1, \dots, x_n\}$ is the linear space generated by $\{x_1, \dots, x_n\}$. By (2.5),

$$\frac{1}{2}U \subset \frac{1}{2}Z + \frac{1}{2^2}U = Z + \frac{1}{2^2}U,$$

so that

$$U \subset Z + \frac{1}{2}U \subset Z + Z + \frac{1}{2^2}U = Z + \frac{1}{2^2}U.$$

Repeating the argument, one obtains

$$U \subset Z + \frac{1}{2^n}U, \tag{2.6}$$

for all $n \in \mathbb{N}$.

We show that $\{\frac{1}{2^n}U\colon n \in \mathbb{N}\}$ is a basis of $\tau(P)$-neighborhoods of 0. For a $\tau(P)$-neighborhood $V$ of 0, there exists $p \in P$ and $r > 0$ such that $B_p(0, r) \subset V$. Since a $P$-precompact set is topologically bounded (with respect to $\tau(P)$), there exists $\lambda > 0$ such that $\lambda U = \lambda B_{p_0}(0, r_0) \subset B_p(0, r)$. If $n \in \mathbb{N}$ is such that $2^{-n} < \lambda$, then

$$\frac{1}{2^n}U = \frac{1}{2^n}B_{p_0}(0, r_0) \subset \lambda B_{p_0}(0, r_0) \subset B_p(0, r) \subset V.$$

It is easy to check that $\{\frac{1}{2^n}B_{\bar{p}_0}(0, r_0)\colon n \in \mathbb{N}\}$ is a basis of $\tau(\bar{P})$-neighborhoods of 0, so that, by Lemma 2.5 and by (2.6),

$$U \subset \bigcap \left\{ Z + \frac{1}{2^n}U\colon n \in \mathbb{N} \right\} = \tau(\bar{P})\text{-}\mathrm{cl}\, Z. \tag{2.7}$$

If we show that $\tau(\bar{P})$-$\mathrm{cl}\, Z = Z$, then by (2.7), for every $x \in X \setminus \{0\}$ there exists $\lambda > 0$ such that $\lambda x \in U \subset Z$, showing that $X = Z$ is finite dimensional.

Let $x \in \tau(\bar{P})$-$\mathrm{cl}\, Z \setminus Z$. Suppose that $\dim Z = m$ and let $e_1, \dots, e_m$ be an algebraic basis of $Z$. The space $W = \mathrm{lin}(Z \cup \{x\})$ has dimension $m + 1$ and $e_1, \dots, e_m, x$ is an algebraic basis of $W$. Since $\{\{\frac{1}{2^n}B_{\bar{p}_0}(0, r_0)\colon n \in \mathbb{N}\}$ is a basis of $\tau(\bar{P})$-neighborhoods of 0, it follows that the topology $\tau(\bar{P})$ is generated by $\bar{p}_0$, so we can work with sequences. Suppose that $z_k = \alpha_{1,k}e_1 + \dots + \alpha_{m,k}e_m + 0 \cdot x$, $k \in \mathbb{N}$, is a sequence in $Z$ which converges to $x = 0 \cdot e_1 + \dots + 0 \cdot e_m + 1 \cdot x$. Since the topology $\tau(P)$ is $T_1$, Proposition 1.4(2), implies that the topology $\tau(\bar{P})$ is also $T_1$. By Lemma 1.9, $\lim_k \alpha_{i,k} = 0$, $i = 1, \dots, m$, and $0 = \lim_k \alpha_{m+1,k} = \alpha_{m+1} = 1$, a contradiction. Consequently, $\tau(\bar{P})$-$\mathrm{cl}\, Z = Z$, and Proposition 2.6 is completely proved. □

The following proposition is the analog of a known result in normed spaces. In the case of asymmetric normed spaces it was proved in [1, Proposition 8].

**Proposition 2.7.** *If $Y$ is a precompact subset of an asymmetric LCS $(X, P)$, then the convex hull $\mathrm{co}\, Y$ of $Y$ is also precompact.*

**Proof.** By Proposition 2.1 it is sufficient to show that $\mathrm{co}\, Y$ is $p$-precompact for every $p \in P$.

Let $p \in P$ and $\epsilon > 0$. By the precompactness of $Y$ there exists a finite subset $\{y_1^0, \dots, y_n^0\}$ of $Y$ such that

$$Y \subset \bigcup_{i=1}^{n} B_p\big(y_i^0, \epsilon\big). \tag{2.8}$$

The set $W = \mathrm{co}\{y_1^0, \dots, y_n^0\}$ is contained in the finite dimensional space $Z = \mathrm{sp}\{y_1^0, \dots, y_n^0\}$ which is isomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}$. The Euclidean norm on $\mathbb{R}^k$ induces, by this isomorphism, a norm $\|\cdot\|$ on $Z$. Since all the norms are equivalent on $Z$, it follows that the set $W$ is compact with respect to the norm $\tilde{p}(\cdot) + \|\cdot\|$, so it is $(\tilde{p}(\cdot) + \|\cdot\|)$-precompact and, a fortiori, $p$-precompact. Let $\{w_1^0, \dots, w_m^0\} \subset W$ be such that

$$W \subset \bigcup_{i=1}^{m} B_p\big(w_i^0, \epsilon\big). \tag{2.9}$$

We show that

$$Y \subset \bigcup_{i=1}^{m} B_p\big(w_i^0, 2\epsilon\big). \tag{2.10}$$

Let $x \in \operatorname{co} Y$, $x = \sum_{i=1}^{l} \alpha_i y_i$ for some $\alpha_i \geqslant 0$, $y_i \in Y$, $i = 1, \dots, l$, $\sum_{i=1}^{l} \alpha_i = 1$. By (2.8), for every $i \in \{1, \dots, l\}$ there exists $j(i) \in \{1, \dots, n\}$ such that $p(y_i - y^0_{j(i)}) \leqslant \epsilon$. Putting $w := \sum_{i=1}^{l} \alpha_i y^0_{j(i)}$, it follows

$$p(x - w) = p\left(\sum_{i=1}^{l} \alpha_i\big(y_i - y^0_{j(i)}\big)\right) \leqslant \sum_{i=1}^{l} \alpha_i p\big(y_i - y^0_{j(i)}\big) \leqslant \epsilon.$$

Since $w \in W$, the equality (2.9) implies the existence of $i_0 \in \{1, \dots, m\}$ such that $p(w - w^0_{i_0}) \leqslant \epsilon$. But then

$$p\big(x - w^0_{i_0}\big) \leqslant p(x - w) + p\big(w - w^0_{i_0}\big) \leqslant 2\epsilon,$$

showing that (2.10) holds. □

Some relations between precompactness and compactness in asymmetric normed spaces were studied in [10] and [1]. We shall extend some of these results to asymmetric LCS.

Let $(X, P)$ be an asymmetric LCS. For $p \in P$ let

$$\theta_{0,p} = \big\{z \in X\colon\ p(z) = 0\big\} \quad \text{and} \quad \theta_0 = \bigcap_{p \in P} \theta_{0,p}.$$

Let also

$$\theta_{x,p} = \big\{z \in X\colon\ p(z - x) = 0\big\} = x + \theta_{0,p}.$$

It is immediate that $\theta_x$ agrees with the $\tau(\bar{P})$-closure of the set $\{x\}$. Indeed

$$\begin{aligned} y \in \tau(\bar{P})\text{-}\mathrm{cl}\{x\} \quad &\iff \quad \forall p \in P,\ \forall \epsilon > 0,\ \bar{p}(x - y) < \epsilon \\ &\iff \quad \forall p \in P,\ \bar{p}(x - y) = 0 \\ &\iff \quad \forall p \in P,\ p(y - x) = 0 \\ &\iff \quad y \in \theta_x. \end{aligned}$$

As it was shown in [10]

$$B'_p(x, \epsilon) = B'_p(x, \epsilon) + \theta_{0,p}.$$

Based on this equality one obtains immediately that

$$Y = Y + \theta_0,$$

for every $\tau(P)$-open subset $Y$ of $X$. Indeed, $0 \in \theta_0$ implies $Y \subset Y + \theta_0$. Conversely, let $x = y + z$ for some $y \in Y$ and $z \in \theta_0$. Since $Y$ is $\tau(P)$-open there exist $p \in P$ and $\epsilon > 0$ such that $B'_p(y, \epsilon) \subset Y$, implying $x = y + z \in B'_p(y, \epsilon) + \theta_0 \subset B'_p(y, \epsilon) + \theta_{0,p} = B'_p(y, \epsilon) \subset Y$.

As a consequence of this last equality, one obtains the analog of Proposition 6 from [10].

**Proposition 2.8.** *A subset $K$ of an asymmetric LCS is $\tau(P)$-compact if and only if $K + \theta_0$ is $\tau(P)$-compact.*

*Also, if $K$ is $\tau(P)$-compact, then every subset $Z$ of $K + \theta_0$ is $\tau(P)$-compact.*

**Remark 2.9.** In the case of an asymmetric normed space $(X, p)$, Alegre et al. [1] give characterizations of $\tau_p$-compact subsets of $X$. Among other results, they prove, under some supplementary hypotheses, that a subset $K$ of $X$ is $\tau_p$-compact if and only if there exists a $\tau_{\bar{p}}$-compact subset $K_0$ of $X$ such that $K_0 \subset K \subset K + \theta_0$ [1, Theorem 20]. It is possible that similar characterizations hold in the locally convex case, a topic for further investigation.

## Acknowledgement

The author expresses his warmest thanks to the reviewer for the pertinent suggestions and remarks that led to an improvement on the paper.

## References


[1] C. Alegre, I. Ferrando, L.M. García-Raffi, E.A. Sánchez-Pérez, Compactness in asymmetric normed spaces, Topology Appl. 155 (6) (2008) 527–539.
[2] S. Cobzaş, Separation of convex sets and best approximation in spaces with asymmetric norm, Quaest. Math. 27 (3) (2004) 275–296.
[3] S. Cobzaş, Asymmetric locally convex spaces, Int. J. Math. Math. Sci. 2005 (16) (2005) 2585–2608.
[4] S. Cobzaş, Compact operators on spaces with asymmetric norm, Stud. Univ. Babes-Bolyai Math. 51 (4) (2006) 69–87.
[5] S. Cobzaş, C. Mustăţa, Extension of bounded linear functionals and best approximation in spaces with asymmetric norm, Rev. Anal. Numer. Theor. Approx. 33 (1) (2004) 39–50.
[6] R. Engelking, General Topology, second ed., Heldermann-Verlag, Berlin, 1989.
[7] J. Ferrer, V. Gregori, A sequentially compact non-compact quasi-pseudometric space, Monatsh. Math. 96 (1983) 269–270.
[8] J. Ferrer, V. Gregori, Completeness and Baire spaces, Math. Chronicle 14 (1985) 39–42.
[9] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, M. Dekker, New York, 1982.
[10] L.M. García-Raffi, Compactness and finite dimension in asymmetric normed linear spaces, Topology Appl. 153 (2005) 844–853.
[11] L.M. García-Raffi, S. Romaguera, E.A. Sánchez-Pérez, The dual space of an asymmetric normed linear space, Quaest. Math. 26 (2003) 83–96.
[12] L.M. García-Raffi, S. Romaguera, E.A. Sánchez-Pérez, On Hausdorff asymmetric normed linear spaces, Houston J. Math. 29 (2003) 717–728.
[13] L.M. García-Raffi, S. Romaguera, E.A. Sánchez-Pérez, Metrizability of the unit ball of the dual of a quasi-normed cone, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 7 (2004) 483–492.
[14] L.M. García-Raffi, S. Romaguera, E.A. Sánchez-Pérez, Weak topologies on asymmetric normed linear spaces and non-asymptotic criteria in the theory of complexity analysis, J. Anal. Appl. 2 (2004) 125–138.
[15] J.L. Kelley, General Topology, Grad. Texts in Math., vol. 27, Springer, New York, 1975.
[16] J.C. Kelly, Bitopological spaces, Proc. London Math. Soc. 13 (1963) 71–89.
[17] H.-P.A. Künzi, A note on sequentially compact quasi-pseudometric spaces, Monatsh. Math. 95 (1983) 219–220.
[18] H.-P.A. Künzi, Quasi-uniform spaces—eleven years later, Topology Proc. 18 (1993) 143–171.
[19] H.-P.A. Künzi, Nonsymmetric topology, in: Topology, Szekszárd, 1993, in: Bolyai Soc. Math. Stud., vol. 4, Janos Bolyai Math. Soc., Budapest, 1995, pp. 303–338.
[20] H.-P.A. Künzi, Nonsymmetric distances and their associated topologies: About the origin of basic ideas in the area of asymmetric topology, in: C.E. Aull, R. Lowen (Eds.), Handbook of the History of General Topology, Kluwer Acad. Publ., Dordrecht, 2001, pp. 853–868.
[21] H.-P.A. Künzi, Quasi-uniform spaces in the year 2001, in: Recent Progress in General Topology, II, North-Holland, Amsterdam, 2002, pp. 313–344.
[22] H.-P.A. Künzi, Uniform structures in the beginning of the third millennium, Topology Appl. 154 (2007) 2745–2756.
[23] H.-P.A. Künzi, M. Mršević, I.L. Reilly, M.K. Vamanamurthy, Convergence, precompactness and symmetry in quasi-uniform spaces, Math. Japon. 38 (1993) 239–253.
[24] P.Th. Lambrinos, On precompact quasi-uniform structures, Proc. Amer. Math. Soc. 62 (1977) 365–366.
[25] R.E. Megginson, An Introduction to Banach Space Theory, Springer, New York, 1998.
[26] M.G. Murdeshwar, S.A. Naimpally, Quasi-Uniform Topological Spaces, Noordhoff, Groningen, 1966.
[27] C. Mustăţa, Extensions of semi-Lipschitz functions on quasi-metric spaces, Rev. Anal. Numer. Theor. Approx. 30 (2001) 61–67.
[28] C. Mustăţa, On the extremal semi-Lipschitz functions, Rev. Anal. Numer. Theor. Approx. 31 (1) (2002) 103–108.
[29] M. Pajoohesh, M.P. Schellekens, A survey of topological work at CEOL, Topology Atlas Invited Contributions 9 (2) (2004), arXiv:math.GN/0412556v1.
[30] I.L. Reilly, P.V. Subrahmanyam, M.K. Vamanamurthy, Cauchy sequences in quasi-pseudo-metric spaces, Monatsh. Math. 93 (1982) 127–140.
[31] S. Romaguera, M. Sanchis, Semi-Lipschitz functions and best approximation in quasi-metric spaces, J. Approx. Theory 103 (2000) 292–301.
[32] S. Romaguera, M. Sanchis, Properties of the normed cone of semi-Lipschitz functions, Acta Math. Hungar. 108 (2005) 55–70.
[33] S. Romaguera, M. Schellekens, Quasi-metric properties of complexity spaces, in: II Iberoamerican Conference on Topology and Its Applications, Morelia, 1997, Topology Appl. 98 (1999) 311–322.
[34] J.L. Sieber, W.J. Pervin, Completeness in quasi-uniform spaces, Math. Ann. 158 (1965) 79–81.